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Characterization of fractional maximal operator and its commutators on Orlicz spaces in the Dunkl setting

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Abstract
On the $\mathbb{R}^d$ the Dunkl operators $\left\{D_{k,j}\right\}_{j=1}^{d}$ are the differential-difference operators associated with the reflection group $\mathbb{Z}_2^d$ on $\mathbb{R}^d$. In this paper, in the setting $\mathbb{R}^d$ we find necessary and sufficient conditions for the boundedness of the fractional maximal operator $M_{\alpha,k}$ on Orlicz spaces $L_{\Phi,k}(\mathbb{R}^d)$. As an application of this result we show that $b \in BMO_k(\mathbb{R}^d)$ if and only if the maximal commutator $M_{b,k}$ is bounded on Orlicz spaces $L_{\Phi,k}(\mathbb{R}^d)$.

Keywords Fractional maximal operator · Orlicz space · Dunkl operator · Commutator · *BMO*

Mathematics Subject Classification 42B20 · 42B25 · 42B35

1 Introduction

It is well known that maximal operators play an important role in harmonic analysis (see [1]). Harmonic analysis associated to the Dunkl transform and the Dunkl differential-difference operator gives rise to convolutions with a relevant generalized

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translation. In the setting $\mathbb{R}^d$ the Dunkl operators $\left\{D_{k,j}\right\}_{j=1}^{d}$, which are the differential-difference operators introduced by Dunkl in [2]. These operators are very important in pure mathematics and in physics. They provide useful tools in the study of special functions with root systems.

Dunkl operators are differential reflection operators associated with finite reflection groups which generalize the usual partial derivatives as well as the invariant differential operators of Riemannian symmetric spaces. They play an important role in harmonic analysis and the study of special functions of several variables. Among other applications, Dunkl operators are employed in the description of quantum integrable models of Calogero-Moser type. Also, there are stochastic processes associated with Dunkl Laplacians which generalize Dyson's Brownian motion model. The Dunkl fractional maximal operator is of particular interest for harmonic analysis associated with root systems. However, the structure of the Dunkl translation makes the study difficult to which the heavy machinery of real analysis cannot be applied, such as covering methods, weighted inequalities, etc.

The harmonic analysis of the Dunkl operator and Dunkl transform was developed in [3–8]. The fractional maximal function, the fractional integral and related topics associated with the Dunkl differential-difference operator have been research areas for many mathematicians such as Abdelkefi and Sifi [9], Deleaval [6], Guliyev and Mammadov [3,4,10], Mammadov [11], Kamoun [12], Mourou [13], Soltani [14,15], Trimeche [16] and others. Moreover, the results on $L_{\Phi,k}(\mathbb{R}^d)$-boundedness of maximal operators associated with D_k were obtained in [10,17].

Norm inequalities for several classical operators of harmonic analysis have been widely studied in the context of Orlicz spaces. It is well known that many of such operators fail to have continuity properties when they act between certain Lebesgue spaces and, in some situations, the Orlicz spaces appear as adequate substitutes. For example, the Hardy-Littlewood maximal operator is bounded on L_p for $1 < p < \infty$, but not on L_1, but using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p = 1$, see [18,19] for more precise statements.

Let T be the classical singular integral operator, the *commutator* $[b, T]$ generated by T and a suitable function b is given by

$$[b, T]f := bT(f) - T(bf). \tag{1.1}$$

A well-known result due to Coifman, Rochberg and Weiss [20] (see also [21]) states that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator $[b, T]$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$.

It is well known that fractional maximal operators play an important role in harmonic analysis (see [1]). Harmonic analysis associated to the Dunkl transform and the Dunkl differential-difference operator gives rise to convolutions with a relevant generalized translation. In this paper, in the framework of this analysis in the setting $\mathbb{R}^d$, we study the boundedness of the fractional maximal commutator $M_{b,\alpha,k}$ and the commutator of the fractional maximal operator, $[b, M_{\alpha,k}]$, on the Orlicz space $L_{\Phi,k}(\mathbb{R}^d)$, when b belongs to the space $BMO_k(\mathbb{R}^d)$, by which some new characterizations of the space $BMO_k(\mathbb{R}^d)$ are given.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries in the Dunkl setting on $\mathbb{R}^d$

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle \cdot, \cdot \rangle$ and its associated norm $\|x\| := \sqrt{\langle x, x \rangle}$ for any $x \in \mathbb{R}^d$. For any $v \in \mathbb{R}^d \backslash \{0\}$ let σ_v be the reflection in the hyperplane $H_v \subset \mathbb{R}^d$ orthogonal to v:

$$\sigma_v(x) := x - \left(\frac{2\langle x, v \rangle}{\|v\|^2} \right) v, \quad \forall x \in \mathbb{R}^d.$$

A finite set $R \subset \mathbb{R}^d \setminus \{0\}$ is called a *root system*, if $\sigma_v R = R$ for all $v \in R$. We assume that it is normalized by $\|v\|^2 = 2$ for all $v \in R$.

The finite group G generated by the reflections $\{\sigma_v\}_{v \in R}$ is called the *reflection group* (or the *Coxeter-Weyl group*) of the root system. Then, we fix a G-invariant function $k : R \to \mathbb{C}$ called the *multiplicity function of the root system* and we consider the family of commuting operators $D_{k,j}$ defined for any $f \in C^1(\mathbb{R}^d)$ and any $x \in \mathbb{R}^d$ by

$$D_{k,j} f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{v \in R_+} k_v \frac{f(x) - f(\sigma_v(x))}{\langle x, v \rangle} \langle v, e_j \rangle, \quad 1 \leq j \leq d,$$

where $C^1(\mathbb{R}^d)$ denotes the set of all functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\left\{ \frac{\partial f}{\partial x_j} \right\}_{j=1}^d$ are continuous on $\mathbb{R}^d$, $\{e_i\}_{i=1}^d$ are the standard unit vectors of $\mathbb{R}^d$ and R_+ is a positive subsystem. These operators, defined by Dunkl [2], are independent of the choice of the positive subsystem R_+ and are of fundamental importance in various areas of mathematics and mathematical physics.

Throughout this paper, we assume that $k_v \geq 0$ for all $v \in R$ and we denote by h_k the weight function on $\mathbb{R}^d$ given by

$$h_k(x) := \prod_{v \in R_+} |\langle x, v \rangle|^{k_v}, \quad \forall x \in \mathbb{R}^d.$$

The function h_k is G-invariant and homogeneous of degree γ_k, where $\gamma_k := \sum_{v \in R_+} k_v$.

Closely related to them is the so-called intertwining operator V_κ (the subscript means that the operator depends on the parameters κ_i, except in the rank-one case where the subscript is then a single parameter). The *intertwining operator* V_κ is the unique linear isomorphism of $\bigoplus_{n \geq 0} P_n$ such that

$$V(P_n) = P_n, V_k(1) = 1, D_i V_k = V_k \frac{\partial}{\partial x_i} \text{ for any } i \in \{1, ..., d\}$$

with P_n being the subspace of homogeneous polynomials of degree n in d variables. The explicit formula of V_k is not known in general (see [22]). For the group $G := \mathbb{Z}_2^d$ and $h_k(x) := \prod_{i=1}^d |x_i|^{k_i}$ for all $x \in \mathbb{R}^d$, it is an integral transform

$$V_k f(x) := b_k \int\limits_{[-1,1]^d} f\big(x_1 t_1, \cdots, x_d t_d\big) \prod_{i=1}^{d} (1+t_i)\Big(1-t_i^2\Big)^{k_i-1} dt, \ \forall x \in \mathbb{R}^d. \tag{2.1}$$

Let $B(x,r) := \{y \in \mathbb{R}^d : |x-y| < r\}$ denote the ball in $\mathbb{R}^d$ that centered in $x \in \mathbb{R}^d$ and having radius $r > 0$. Then having

$$|B(0,r)|_k = \int_{B(0,r)} h_k^2(x)dx = \left(\frac{a_k}{d+2\gamma_k}\right) r^{d+2\gamma_k},$$

where

$$a_k := \left(\int_{S^{d-1}} h_k^2(x)\, d\sigma(x)\right)^{-1},$$

S^{d-1} is the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$.

The *fractional maximal operator* $M_{\alpha,k}$, $0 < \alpha < d+2\gamma_k$ associated with the Dunkl operator on $\mathbb{R}^d$ is given by (see [6])

$$M_k f(x) := \sup_{r>0} \Big(|B(x,r)|_k\Big)^{-1+\frac{\alpha}{d+2\gamma_k}} \int_{B(x,r)} |f(y)|\, h_k^2(y)dy, \quad x \in \mathbb{R}^d$$

and the *fractional maximal commutator* $M_{b,\alpha,k}$, $0 < \alpha < d+2\gamma_k$ associated with the Dunkl operator on $\mathbb{R}^d$ and with a locally integrable function $b \in L_{1,k}^{\mathrm{loc}}(\mathbb{R}^d) \equiv L_1^{\mathrm{loc}}(\mathbb{R}^d, h_k^2(x)dx)$ is defined by (see [23])

$$M_{b,\alpha,k} f(x) := \sup_{r>0} \Big(|B(x,r)|_k\Big)^{-1+\frac{\alpha}{d+2\gamma_k}}$$
$$\int_{B(x,r)} |b(x)-b(y)|\, |f(y)|\, h_k^2(y)dy, \quad \forall x \in \mathbb{R}^d.$$

If $\alpha = 0$, then $M_k \equiv M_{k,0}$ is the maximal operator associated by Dunkl operator on $\mathbb{R}^d$ and $M_{b,k} \equiv M_{b,k,0}$ is the maximal commutator operator associated by Dunkl operator on $\mathbb{R}^d$.

On the other hand, similar to (1.1), we can define the (nonlinear) commutator of the fractional maximal operator $M_{\alpha,k}$ with a locally integrable function b by

$$[b, M_{\alpha,k}](f)(x) f = b(x) M_{\alpha,k}(f)(x) - M_{\alpha,k}(bf)(x).$$

For more details about the operators $M_{b,k}$ and $[b, M_k]$, we refer to [10] and references therein.

For a function b defined on $\mathbb{R}$, we let, for any $x \in \mathbb{R}^d$,

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \geq 0, \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, for any $x \in \mathbb{R}$, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_{\alpha,k}]$ and $M_{b,\alpha,k}$ are valid:

Let b be any non-negative locally integrable function. Then

$$|[b, M_{\alpha,k}]f(x)| \leq M_{b,\alpha,k}(f)(x), \quad \forall\, x \in \mathbb{R}^d$$

holds for all $f \in L_1^{\text{loc}}(\mathbb{R}^d, h_k^2(x)dx)$.

If b is any locally integrable function on $\mathbb{R}$, then

$$|[b, M_{\alpha,k}]f(x)| \leq M_{b,\alpha,k}(f)(x) + 2b^-(x)M_{\alpha,k}f(x), \quad \forall\, x \in \mathbb{R}^d \tag{2.2}$$

holds for all $f \in L_{1,k}^{\text{loc}}(\mathbb{R}^d)$ (see, for example, [24]).

2.1 Orlicz spaces in the Dunkl setting on $\mathbb{R}^d$

Recall that Orlicz space was first introduced by Orlicz in [25,26] as a generalizations of Lebesgue spaces L_p. Since then this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz space is also an appropriate substitute for L_1 space when the space L_1 does not work.

To introduce the notion of Orlicz spaces in the Dunkl setting on $\mathbb{R}^d$, we first recall the definition of Young functions.

Definition 1 A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, \infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \quad \text{for} \quad 0 < r < \infty$$

is denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) := \inf\{r \geq 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for any } r \ge 0, \tag{2.3}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) := \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty) \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-*condition*, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\,\Phi(r), \quad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-*condition*, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2C}\Phi(Cr), \quad r \ge 0$$

for some $C > 1$. In what follows, for any subset E of $\mathbb{R}$, we use χ_E to denote its *characteristic function*.

Definition 2 (*Orlicz space*) For a Young function Φ, the set

$$L_{\Phi,k}(\mathbb{R}^d) \equiv L_\Phi(\mathbb{R}^d, h_k^2(x)dx) = \left\{ f \in L_{1,k}^{\text{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \Phi(\lambda|f(x)|)\, h_k^2(x)dx < \infty \text{ for some } \lambda > 0 \right\}$$

is called the *Orlicz space*. If $\Phi(r) := r^p$ for all $r \in [0, \infty)$, $1 \le p < \infty$, then $L_{\Phi,k}(\mathbb{R}^d) = L_{p,k}(\mathbb{R}^d) \equiv L_p(\mathbb{R}^d, h_k^2(x)dx)$. If $\Phi(r) := 0$ for all $r \in [0, 1]$ and $\Phi(r) := \infty$ for all $r \in (1, \infty)$, then $L_{\Phi,k}(\mathbb{R}^d) = L_{\infty,k}(\mathbb{R}^d)$. The *space* $L_{\Phi,k}^{\text{loc}}(\mathbb{R}^d)$ is defined as the set of all functions f such that $f\chi_B \in L_{\Phi,k}(\mathbb{R}^d)$ for all balls $B \subset \mathbb{R}^d$.

$L_{\Phi,k}(\mathbb{R}^d)$ is a Banach space with respect to the norm

$$\|f\|_{L_{\Phi,k}} := \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) h_k^2(x)dx \le 1 \right\}.$$

For a measurable function f on $\mathbb{R}^d$ and $t > 0$, let

$$m(f, t)_k := |\{x \in \mathbb{R}^d : |f(x)| > t\}|_k.$$

Definition 3 The *weak Orlicz space*

$$WL_{\Phi,k}(\mathbb{R}^d) := \left\{ f \in L_1^{\text{loc}}(\mathbb{R}^d, h_k^2(x)dx) : \|f\|_{WL_{\Phi,k}} < \infty \right\}$$

is defined by the norm

$$\|f\|_{WL_{\Phi,k}} := \inf\Big\{\lambda > 0 \,:\, \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\, t\Big)_k \le 1\Big\}.$$

We note that $\|f\|_{WL_{\Phi,k}} \le \|f\|_{L_{\Phi,k}}$,

$$\sup_{t>0} \Phi(t) m(f,t)_k = \sup_{t>0} t\, m(f, \Phi^{-1}(t))_k = \sup_{t>0} t\, m(\Phi(|f|), t)_k$$

and

$$\int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\|f\|_{L_{\Phi,k}}}\right) h_k^2(x)dx \le 1, \quad \sup_{t>0} \Phi(t) m\left(\frac{f}{\|f\|_{WL_{\Phi,k}}},\, t\right)_k \le 1. \tag{2.4}$$

The following analogue of the Hölder inequality is well known (see, for example, [27]).

Theorem 1 *Let the functions f and g be measurable on $\mathbb{R}^d$. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_{\mathbb{R}^d} |f(x)g(x)|\, h_k^2(x)dx \le 2\|f\|_{L_{\Phi,k}} \|g\|_{L_{\widetilde{\Phi},k}}.$$

By elementary calculations we have the following property.

Lemma 1 *Let Φ be a Young function and B be a ball in $\mathbb{R}^d$. Then*

$$\|\chi_B\|_{L_{\Phi,k}} = \|\chi_B\|_{WL_\Phi} = \frac{1}{\Phi^{-1}\Big(|B|_k^{-1}\Big)}.$$

By Theorem 1, Lemma 1 and (2.3) we obtain the following estimate.

Lemma 2 *For a Young function Φ and for the ball B the following inequality is valid:*

$$\int_B |f(y)|\, h_k^2(x)dx \le 2|B|_k\, \Phi^{-1}\Big(|B|_k^{-1}\Big) \|f\,\chi_B\|_{L_{\Phi,k}}.$$

We begin with the boundedness of the maximal operator M_k on Orlicz spaces $L_{\Phi,k}(\mathbb{R}^d)$, which were proved in [10,17], see also [28].

Theorem 2 [10,17] *Let Φ be a Young function.*

(i) *The operator M_k is bounded from $L_{\Phi,k}(\mathbb{R}^d)$ to $WL_{\Phi,k}(\mathbb{R}^d)$, and the inequality*

$$\|M_k f\|_{WL_{\Phi,k}} \le C_0 \|f\|_{L_{\Phi,k}} \tag{2.5}$$

holds with constant C_0 independent of f.

(ii) *The operator M_k is bounded on $L_{\Phi,k}(\mathbb{R}^d)$, and the inequality*

$$\|M_k f\|_{L_{\Phi,k}} \le C_0 \|f\|_{L_{\Phi,k}} \tag{2.6}$$

holds with constant C_0 independent of f if and only if $\Phi \in \nabla_2$.

The following theorems were proved in [10,17].

Theorem 3 [10] *Let $b \in BMO_k(\mathbb{R}^d)$ and $\Phi \in \mathcal{Y}$. Then the condition $\Phi \in \nabla_2$ is necessary and sufficient for the boundedness of $M_{b,k}$ on $L_{\Phi,k}(\mathbb{R}^d)$.*

Theorem 4 [10] *Let Φ be a Young function with $\Phi \in \nabla_2$. Then the condition $b \in BMO_k(\mathbb{R}^d)$ is necessary and sufficient for the boundedness of $M_{b,k}$ on $L_{\Phi,k}(\mathbb{R}^d)$.*

From (2.2) and Theorem 4 we deduce the following conclusion.

Corollary 1 *Let Φ be a Young function with $\Phi \in \nabla_2$. Then the conditions $b^+ \in BMO_k(\mathbb{R}^d)$ and $b^- \in L_{\infty,k}(\mathbb{R}^d)$ are sufficient for the boundedness of $[b, M_k]$ on $L_{\Phi,k}(\mathbb{R}^d)$.*

3 Fractional maximal operator $M_{\alpha,k}$ in Orlicz spaces $L_{\Phi,k}(\mathbb{R}^d)$

In this section, we shall give a necessary and sufficient condition for the boundedness of $M_{\alpha,k}$ on Orlicz spaces $L_{\Phi,k}(\mathbb{R}^d)$ and weak Orlicz spaces $WL_{\Phi,k}(\mathbb{R}^d)$.

In order to prove our main theorem, we also need the following lemma.

Lemma 3 *If $B_0 := B(x_0, r_0)$, then*

$$|B_0|_k^{\frac{\alpha}{d+2\gamma_k}} \le M_{\alpha,k}\chi_{B_0}(x)$$

for every $x \in B_0$.

Proof For $x \in B_0$, we get

$$\begin{aligned} M_{\alpha,k}\chi_{B_0}(x) &= \sup_{B \ni x} |B_0|_k^{-1+\frac{\alpha}{d+2\gamma_k}} |B \cap B_0|_k \\ &\ge |B_0|_k^{-1+\frac{\alpha}{d+2\gamma_k}} |B_0 \cap B_0|_k = |B_0|_k^{\frac{\alpha}{d+2\gamma_k}}. \end{aligned}$$

□

The following result completely characterizes the boundedness of $M_{\alpha,k}$ on Orlicz spaces $L_{\Phi,k}(\mathbb{R}^d)$.

Theorem 5 *Let $0 < \alpha < d+2\gamma_k$, Φ, Ψ be Young functions and $\Phi \in \mathcal{Y}$. The condition*

$$r^{-\frac{\alpha}{d+2\gamma_k}} \Phi^{-1}(r) \le C\Psi^{-1}(r) \tag{3.1}$$

for all $r > 0$, where $C > 0$ does not depend on r, is necessary and sufficient for the boundedness of $M_{\alpha,k}$ from $L_{\Phi,k}(\mathbb{R}^d)$ to $WL_{\Psi,k}(\mathbb{R}^d)$. Moreover, if $\Phi \in \nabla_2$, the condition (3.1) is necessary and sufficient for the boundedness of $M_{\alpha,k}$ from $L_{\Phi,k}(\mathbb{R}^d)$ to $L_{\Psi,k}(\mathbb{R}^d)$.

Proof For arbitrary ball $B = B(x, r)$ we represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \quad r > 0,$$

and have

$$M_{\alpha,k} f(x) = M_{\alpha,k} f_1(x) + M_{\alpha,k} f_2(x).$$

Let y be an arbitrary point in B. If $B(y, t) \cap {}^{\complement}(B(x, 2r)) \neq \emptyset$, then $t > r$. Indeed, if $z \in B(y, t) \cap {}^{\complement}(B(x, 2r))$, then $t > |y - z| \geq |x - z| - |x - y| > 2r - r = r$.

On the other hand, $B(y, t) \cap {}^{\complement}(B(x, 2r)) \subset B(x, 2t)$. Indeed, if $z \in B(y, t) \cap {}^{\complement}(B(x, 2r))$, then we get $|x - z| \leq |y - z| + |x - y| < t + r < 2t$.

Hence

$$\begin{aligned} M_{\alpha,k} f_2(y) &\lesssim \sup_{t>0} \frac{1}{|B(y,t)|_k^{1-\frac{\alpha}{d+2\gamma_k}}} \int_{B(y,t)\cap {}^{\complement}(B(x,2r))} |f(z)|\, h_k^2(z)dz \\ &\lesssim \sup_{t>r} \frac{1}{|B(x,2t)|_k^{1-\frac{\alpha}{d+2\gamma_k}}} \int_{B(x,2t)} |f(z)|\, h_k^2(z)dz \\ &= \sup_{t>2r} \frac{1}{|B(x,r)|_k^{1-\frac{\alpha}{d+2\gamma_k}}} \int_{B(x,t)} |f(z)|\, h_k^2(z)dz \\ &\lesssim \sup_{r<t<\infty} t^{\alpha}\, \Phi^{-1}(|B(x,r)|_k^{-1})\, \|f\|_{L_{\Phi,k}(B(x,t))} \\ &\lesssim \|f\|_{L_{\Phi,k}} \sup_{r<t<\infty} t^{\alpha}\, \Phi^{-1}(|B(x,r)|_k^{-1}). \end{aligned}$$

Consequently from Hedberg's trick, see [29], and the last inequality, we have

$$M_{\alpha,k} f(y) \lesssim r^{\alpha} M_k f(y) + \|f\|_{L_{\Phi,k}} \sup_{r<t<\infty} t^{\alpha}\, \Phi^{-1}(t^{-d-2\gamma_k}).$$

Thus, by (3.1) we obtain

$$M_{\alpha,k} f(x) \lesssim M_k f(x) \frac{\Psi^{-1}(r^{-d-2\gamma_k})}{\Phi^{-1}(r^{-d-2\gamma_k})} + \|f\|_{L_{\Phi,k}}\, \Psi^{-1}(r^{-d-2\gamma_k}).$$

Choose $r > 0$ so that $\Phi^{-1}(r^{-d-2\gamma_k}) = \frac{M_k f(x)}{C_0 \|f\|_{L_{\Phi,k}}}$. Then

$$\frac{\Psi^{-1}(r^{-d-2\gamma_k})}{\Phi^{-1}(r^{-d-2\gamma_k})} = \frac{(\Psi^{-1} \circ \Phi)(\frac{M_k f(x)}{C_0 \|f\|_{L_{\Phi,k}}})}{\frac{M_k f(x)}{C_0 \|f\|_{L_{\Phi,k}}}}.$$

Therefore, we get

$$M_{\alpha,k} f(x) \leq C_1 \|f\|_{L_{\Phi,k}} (\Psi^{-1} \circ \Phi)\Big(\frac{M_k f(x)}{C_0 \|f\|_{L_{\Phi,k}}}\Big).$$

Let C_0 be as in (2.5). Then by Theorem 2, we have

$$\begin{aligned}
&\sup_{r>0} \Psi(r)\, m\left(B, \frac{M_{\alpha,k} f(x)}{C_1 \|f\|_{L_{\Phi,k}}}, r\right)_k \\
&\quad = \sup_{r>0} r\, m\left(B, \Psi\left(\frac{M_{\alpha,k} f(x)}{C_1 \|f\|_{L_{\Phi,k}}}\right), r\right)_k \\
&\quad \leq \sup_{r>0} r\, m\left(B, \Phi\left(\frac{M_k f(x)}{C_0 \|f\|_{L_{\Phi,k}}}\right), r\right)_k \leq \sup_{r>0} \Phi(r)\, m\left(\frac{Mf(x)}{\|Mf\|_{WL_{\Phi,k}}}, r\right)_k \leq 1,
\end{aligned}$$

i.e.

$$\|M_{\alpha,k} f\|_{WL_{\Psi,k}(B)} \lesssim \|f\|_{L_{\Phi,k}}. \tag{3.2}$$

By taking supremum over B in (3.2), we get

$$\|M_{\alpha,k} f\|_{WL_{\Psi,k}} \lesssim \|f\|_{L_{\Phi,k}},$$

since the constants in (3.2) don't depend on x and r.

Let C_0 be as in (2.6). Since $\Phi \in \nabla_2$, by Theorem 2, we have

$$\begin{aligned}
\int_B \Psi\left(\frac{M_{\alpha,k} f(x)}{C_1 \|f\|_{L_{\Phi,k}}}\right) h_k^2(x)dx &\leq \int_B \Phi\left(\frac{M_k f(x)}{C_0 \|f\|_{L_{\Phi,k}}}\right) h_k^2(x)dx \\
&\leq \int_{\mathbb{R}^n} \Phi\left(\frac{Mf(x)}{\|M_k f\|_{L_{\Phi,k}}}\right) dx \leq 1,
\end{aligned}$$

i.e.

$$\|M_{\alpha,k} f\|_{L_{\Psi,k}(B)} \lesssim \|f\|_{L_{\Phi,k}}. \tag{3.3}$$

By taking supremum over B in (3.3), we get

$$\|M_{\alpha,k} f\|_{L_{\Psi,k}} \lesssim \|f\|_{L_{\Phi,k}},$$

since the constants in (3.3) don't depend on x and r.

We shall now prove the necessity. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 3, we have $r_0^\alpha \le C M_{\alpha,k}\chi_{B_0}(x)$. Therefore, by Lemma 1, we have

$$r_0^\alpha \lesssim \Psi^{-1}(|B_0|_k^{-1}) \|M_{\alpha,k}\chi_{B_0}\|_{WL_{\Psi,k}(B_0)} \lesssim \Psi^{-1}(|B_0|_k^{-1}) \|M_{\alpha,k}\chi_{B_0}\|_{WL_{\Psi,k}}$$
$$\lesssim \Psi^{-1}(|B_0|_k^{-1}) \|\chi_{B_0}\|_{L_{\Phi,k}} \lesssim \frac{\Psi^{-1}(r_0^{-d-2\gamma_k})}{\Phi^{-1}(r_0^{-d-2\gamma_k})}$$

and

$$r_0^\alpha \lesssim \Psi^{-1}(|B_0|_k^{-1}) \|M_{\alpha,k}\chi_{B_0}\|_{L_{\Psi,k}(B_0)} \lesssim \Psi^{-1}(|B_0|_k^{-1}) \|M_{\alpha,k}\chi_{B_0}\|_{L_{\Psi,k}}$$
$$\lesssim \Psi^{-1}(|B_0|_k^{-1}) \|\chi_{B_0}\|_{L_{\Phi,k}} \lesssim \frac{\Psi^{-1}(r_0^{-d-2\gamma_k})}{\Phi^{-1}(r_0^{-d-2\gamma_k})}.$$

Since this is true for every $r_0 > 0$, we are done. □

We recover the following well known result by taking $\Phi(t) = t^p$ at Theorem 5.

Corollary 2 *Let $0 < \alpha < d + 2\gamma_k$ and $1 \le p < (d + 2\gamma_k)/\alpha$. Then the condition $1/q = 1/p - \alpha/(d + 2\gamma_k)$ is necessary and sufficient for the boundedness of $M_{\alpha,k}$ from $L_{p,k}(\mathbb{R}^d)$ to $WL_{q,k}(\mathbb{R}^d)$ and for $p > 1$ from $L_{p,k}(\mathbb{R}^d)$ to $L_{q,k}(\mathbb{R}^d)$.*

4 Fractional maximal commutator $M_{b,\alpha,k}$ in Orlicz spaces $L_{\Phi,k}(\mathbb{R}^d)$

In this section we investigate the boundedness of the fractional maximal commutator $M_{b,\alpha,k}$ and the commutator of the fractional maximal operator, $[b, M_{\alpha,k}]$, in Orlicz spaces $L_{\Phi,k}(\mathbb{R}^d)$.

We recall the definition of the space $BMO_k(\mathbb{R}^d)$.

Definition 4 Suppose that $b \in L_{1,k}^{\text{loc}}(\mathbb{R}^d)$, let

$$\|b\|_{BMO_k} := \sup_{x\in\mathbb{R}, r>0} \frac{1}{|B(x,r)|_k} \int_{B(x,r)} |b(y) - b_{B(x,r)}(x)| \; h_k^2(y)dy,$$

where

$$b_{B(x,r)} := \frac{1}{|B(x,r)|_k} \int_{B(x,r)} b(y) \; h_k^2(y)dy.$$

Define

$$BMO_k(\mathbb{R}^d) := \{b \in L_{1,k}^{\text{loc}}(\mathbb{R}^d) \; : \; \|b\|_{BMO_k} < \infty\}.$$

Modulo constants, the space $BMO_k(\mathbb{R}^d)$ is a Banach space with respect to the norm $\|\cdot\|_{BMO_k(\mathbb{R}^d)}$.

We will need the following properties of BMO-functions (see [21]):

$$\|b\|_{BMO_k} \approx \sup_{x\in\mathbb{R}^d, r>0} \left(\frac{1}{|B(x,r)|_k} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p \, h_k^2(y)dy \right)^{\frac{1}{p}}, \tag{4.1}$$

where $1 \le p < \infty$ and the positive equivalence constants are independent of b, and

$$\left| b_{B(x,r)} - b_{B(x,t)} \right| \le C\|b\|_{BMO_k} \, \ln\frac{t}{r} \quad \text{for any} \quad 0 < 2r < t, \tag{4.2}$$

where the positive constant C does not depend on b, x, r and t.

Next, we recall the notion of weights. Let w be a locally integrable and positive function on $(\mathbb{R}^d, h_k^2(x)dx)$. The function w is called a *Muckenhoupt* $A_{1,k}(\mathbb{R}^d)$ *weight* if there exists a positive constant C such that for any ball B

$$\frac{1}{|B|_k} \int_B w(x)\, h_k^2(x)dx \le C \operatorname*{ess\,inf}_{x\in B} w(x).$$

Lemma 4 [28, Chapter 1] *Let* $\omega \in A_{1,k}(\mathbb{R}^d)$*, then the reverse Hölder inequality holds, that is, there exist* $q > 1$ *and a positive constant* C *such that*

$$\left(\frac{1}{|B|_k} \int_B w(x)^q \, h_k^2(x)dx \right)^{\frac{1}{q}} \le \frac{C}{|B|_k} \int_B w(x)\, h_k^2(x)dx$$

for all balls B.

Lemma 5 [10] *Let* Φ *be a Young function with* $\Phi \in \Delta_2$*,* B *be a ball in* $\mathbb{R}^d$ *and* $f \in L_{\Phi,k}(B)$*. Then we have*

$$\frac{1}{2|B|_k} \int_B |f(x)|\, h_k^2(x)dx \le \Phi^{-1}\big(|B|_k^{-1}\big) \; \|f\|_{L_{\Phi,k}}$$
$$\le C \left(\frac{1}{|B|_k} \int_B |f(x)|^p \, h_k^2(x)dx \right)^{\frac{1}{p}}$$

for some $1 < p < \infty$*, where the positive constant* C *does not depend on* f *and* B.

We have the following result from (4.1) and Lemma 5.

Lemma 6 *Let* $b \in BMO_k(\mathbb{R}^d)$ *and* Φ *be a Young function with* $\Phi \in \Delta_2$*, then*

$$\|b\|_{BMO_k} \approx \sup_{x\in\mathbb{R}^d, r>0} \Phi^{-1}\big(|B(x,r)|_k^{-1}\big) \left\| b(\cdot) - b_{B(x,r)} \right\|_{L_{\Phi,k}(B(x,r))}, \tag{4.3}$$

where the positive equivalence constants are independent of b.

By Theorem 2 and Theorem 1.13 in [24] we obtain the following theorem.

Theorem 6 [10] *Let $b \in BMO_k(\mathbb{R}^d)$ and and Φ be a Young function. Then the condition $\Phi \in \nabla_2$ is necessary and sufficient for the boundedness of $M_{b,k}$ on $L_{\Phi,k}(\mathbb{R}^d)$, i.e., the inequality*

$$\|M_{b,k} f\|_{L_{\Phi,k}} \leq C_0 \|b\|_{BMO_k} \|f\|_{L_{\Phi,k}} \tag{4.4}$$

holds with constant C_0 independent of f.

The following lemma is the analogue of the Hedberg's trick for the commutator of fractional integral (see [29]).

Lemma 7 *If $0 < \alpha < d + 2\gamma_k$ and $f, b \in L^{\mathrm{loc}}_{1,k}(\mathbb{R}^d)$, then for all $x \in \mathbb{R}^d$ and $r > 0$ we get*

$$\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{d+2\gamma_k-\alpha}} |b(x)-b(y)|\, h_k^2(y)dy \lesssim r^\alpha M_{b,k} f(x).$$

Proof

$$\begin{aligned}
&\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{d+2\gamma_k-\alpha}} |b(x)-b(y)|\, h_k^2(y)dy \\
&= \sum_{j=0}^{\infty} \int_{2^{-j-1}r \leq |x-y| < 2^{-j}r} \frac{|f(y)|}{|x-y|^{d+2\gamma_k-\alpha}} |b(x)-b(y)|\, h_k^2(y)dy \\
&\lesssim \sum_{j=0}^{\infty} (2^{-j}r)^\alpha (2^{-j}r)^{-1} \int_{|x-y|<2^{-j}r} |f(y)||b(x)-b(y)|\, h_k^2(y)dy \\
&\lesssim r^\alpha M_{b,k} f(x).
\end{aligned}$$

□

For proving our main results, we need the following estimate.

Lemma 8 *If $b \in L^{\mathrm{loc}}_{1,k}(\mathbb{R}^d)$ and $B_0 := B(x_0, r_0)$, then*

$$r_0^\alpha |b(x) - b_{B_0}| \leq C M_{b,\alpha,k} \chi_{B_0}(x) \ \text{ for every } \ x \in B_0.$$

Proof It is well known that

$$\mathrm{M}_{b,\nu,\alpha} f(x) \leq 2^{d+2\gamma_k-\alpha} M_{b,\alpha,k} f(x), \tag{4.5}$$

where $\mathrm{M}_{b,\nu,\alpha}(f)(x) = \sup_{B \ni x} |B|_k^{-d-2\gamma_k+\alpha} \int_B |b(x)-b(y)||f(y)|\, h_k^2(y)dy$.

Now let $x \in B_0$. By using (4.5), we get

$$\begin{aligned}
M_{b,\alpha,k} \chi_{B_0}(x) &\geq C \mathrm{M}_{b,\nu,\alpha} f(x) \\
&= C \sup_{B \ni x} |B|_k^{-d-2\gamma_k+\alpha} \int_B |b(x)-b(y)| \chi_{B_0}\, h_k^2(y)dy \\
&= C \sup_{B \ni x} |B|_k^{-d-2\gamma_k+\alpha} \int_{B \cap B_0} |b(x)-b(y)|\, h_k^2(y)dy
\end{aligned}$$

$$\geq C|B_0|_k^{-d-2\gamma_k+\alpha} \int_{B_0 \cap B_0} |b(x)-b(y)|\, h_k^2(y)dy$$
$$\geq C|B_0|_k^{-d-2\gamma_k+\alpha} \Big| \int_{B_0} (b(x)-b(y))\, h_k^2(y)dy \Big|$$
$$= Cr_0^{\alpha}\, |b(x)-b_{B_0}|.$$

□

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $M_{b,\alpha,k}$ from $L_{\Phi,k}(\mathbb{R}^d)$ to $L_{\Psi,k}(\mathbb{R}^d)$.

Theorem 7 *Let* $0 < \alpha < d+2\gamma_k$, $b \in BMO_k(\mathbb{R}^d)$ *and* Φ, Ψ *be Young functions and* $\Phi \in \mathcal{Y}$.

1. *If* $\Phi \in \nabla_2$ *and* $\Psi \in \Delta_2$, *then the condition*

$$r^{\alpha}\Phi^{-1}\big(r^{-d-2\gamma_k}\big) + \sup_{r<t<\infty} \Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-d-2\gamma_k}\big)t^{\alpha} \leq C\Psi^{-1}\big(r^{-d-2\gamma_k}\big) \quad (4.6)$$

for all $r > 0$, *where* $C > 0$ *does not depend on* r, *is sufficient for the boundedness of* $M_{b,\alpha,k}$ *from* $L_{\Phi,k}(\mathbb{R}^d)$ *to* $L_{\Psi}(\mathbb{R}^d, h_k^2(x)dx)$.

2. *If* $\Psi \in \Delta_2$, *then the condition* (3.1) *is necessary for the boundedness of* $M_{b,\alpha,k}$ *from* $L_{\Phi,k}(\mathbb{R}^d)$ *to* $L_{\Psi,k}(\mathbb{R}^d)$.
3. *Let* $\Phi \in \nabla_2$ *and* $\Psi \in \Delta_2$. *If the condition*

$$\sup_{r<t<\infty} \Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-d-2\gamma_k}\big)t^{\alpha} \leq Cr^{\alpha}\Phi^{-1}\big(r^{-d-2\gamma_k}\big) \quad (4.7)$$

holds for all $r > 0$, *where* $C > 0$ *does not depend on* r, *then the condition* (3.1) *is necessary and sufficient for the boundedness of* $M_{b,\alpha,k}$ *from* $L_{\Phi,k}(\mathbb{R}^d)$ *to* $L_{\Psi,k}(\mathbb{R}^d)$.

Proof 1. For arbitrary $x_0 \in \mathbb{R}$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$.

Let x be an arbitrary point in B. If $B(x,t) \cap \{\complement(2B)\} \neq \emptyset$, then $t > r$. Indeed, if $y \in B(x,t) \cap \{\complement(2B)\}$, then $t > |x-y| \geq |x_0-y| - |x_0-x| > 2r - r = r$.

On the other hand, $B(x,t) \cap \{\complement(2B)\} \subset B(x_0, 2t)$. Indeed, if $y \in B(x,t) \cap \{\complement(2B)\}$, then we get $|x_0-y| \leq |x-y| + |x_0-x| < t+r < 2t$.

Hence

$$M_{b,\alpha,k}(f_2)(x) = \sup_{t>0} \frac{1}{|B(x,t)|_k^{1-\frac{\alpha}{d+2\gamma_k}}} \int_{B(x,t)\cap \complement(2B)} |b(y)-b(x)||f(y)|\, h_k^2(y)dy$$
$$\leq 2^{n-\alpha} \sup_{t>r} \frac{1}{|B(x_0,2t)|_k^{1-\frac{\alpha}{d+2\gamma_k}}} \int_{B(x_0,2t)} |b(y)-b(x)||f(y)|\, h_k^2(y)dy$$
$$= 2^{n-\alpha} \sup_{t>2r} \frac{1}{|B(x_0,t)|_k^{1-\frac{\alpha}{d+2\gamma_k}}} \int_{B(x_0,t)} |b(y)-b(x)||f(y)|\, h_k^2(y)dy.$$

Therefore, for all $x \in B$ we have

$$\begin{aligned}
M_{b,\alpha,k}(f_2)(x) &\lesssim \sup_{t>2r} t^{\alpha-d-2\gamma_k} \int_{B(x_0,t)} |b(y)-b(x)||f(y)|\, h_k^2(y)dy \\
&\lesssim \sup_{t>2r} t^{\alpha-d-2\gamma_k} \int_{B(x_0,t)} |b(y)-b_{B(x_0,t)}||f(y)|\, h_k^2(y)dy \\
&\quad + \sup_{t>2r} t^{\alpha-d-2\gamma_k} \int_{B(x_0,t)} |b_{B(x_0,t)}-b_B||f(y)|\, h_k^2(y)dy \\
&\quad + \sup_{t>2r} t^{\alpha-d-2\gamma_k} \int_{B(x_0,t)} |b_B-b(x)||f(y)|\, h_k^2(y)dy \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

Applying Hölder's inequality, by (2.3), (4.2), (4.3) and Lemma 2 we get

$$\begin{aligned}
J_1 + J_2 &\lesssim \sup_{t>2r} t^{\alpha-d-2\gamma_k} \int_{B(x_0,t)} |b(y)-b_{B(x_0,t)}||f(y)|\, h_k^2(y)dy \\
&\quad + \sup_{t>2r} t^{\alpha-d-2\gamma_k} |b_{B(x_0,r)}-b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)|\, h_k^2(y)dy \\
&\lesssim \sup_{t>2r} t^{\alpha-d-2\gamma_k} \left\| b(\cdot)-b_{B(x_0,t)} \right\|_{L_{\widetilde{\Phi}}(B(x_0,t))} \|f\|_{L_{\Phi,k}(B(x_0,t))} \\
&\quad + \sup_{t>2r} t^{\alpha-d-2\gamma_k} |b_{B(x_0,r)}-b_{B(x_0,t)}| t^{d+2\gamma_k} \Phi^{-1}\big(|B(x_0,t)|_k^{-1}\big) \|f\|_{L_{\Phi,k}(B(x_0,t))} \\
&\lesssim \|b\|_{BMO_k} \sup_{t>2r} \Phi^{-1}(|B(x_0,t)|_k^{-1})\, t^{\alpha} \left(1+\ln\frac{t}{r}\right) \|f\|_{L_{\Phi,k}(B(x_0,t))} \\
&\lesssim \|b\|_{BMO_k} \|f\|_{L_{\Phi,k}} \sup_{t>2r} \left(1+\ln\frac{t}{r}\right) t^{\alpha}\, \Phi^{-1}(t^{-d-2\gamma_k}).
\end{aligned}$$

A geometric observation shows $2B \subset B(x, 3r)$ for all $x \in B$. Using Lemma 7, we get

$$\begin{aligned}
J_0(x) := M_{b,\alpha,k}(f_1)(x) \lesssim |b, I_\alpha|(|f_1|)(x) &= \int_{2B} \frac{|b(y)-b(x)|}{|x-y|^{d+2\gamma_k-\alpha}} |f(y)|\, h_k^2(y)dy \\
\lesssim \int_{B(x,3r)} \frac{|b(y)-b(x)|}{|x-y|^{d+2\gamma_k-\alpha}} |f(y)|\, h_k^2(y)dy &\lesssim r^{\alpha} M_{b,k} f(x).
\end{aligned}$$

Consequently for all $x \in B$ we get

$$\begin{aligned}
J_0(x) + J_1 + J_2 &\lesssim \|b\|_{BMO_k} r^{\alpha} M_{b,k} f(x) \\
&\quad + \|b\|_{BMO_k} \|f\|_{L_{\Phi,k}} \sup_{t>2r} \left(1+\ln\frac{t}{r}\right) t^{\alpha} \Phi^{-1}(t^{-d-2\gamma_k}).
\end{aligned}$$

Thus, by (4.6) we obtain

$$J_0(x) + J_1 + J_2 \lesssim \|b\|_{BMO_k} \left(M_{b,k} f(x) \frac{\Psi^{-1}(r^{-d-2\gamma_k})}{\Phi^{-1}(r^{-d-2\gamma_k})} + \Psi^{-1}(r^{-d-2\gamma_k}) \|f\|_{L_{\Phi,k}} \right).$$

Choose $r > 0$ so that $\Phi^{-1}(r^{-d-2\gamma_k}) = \frac{M_{b,k}f(x)}{C_0\|b\|_{BMO_k}\|f\|_{L_{\Phi,k}}}$. Then

$$\frac{\Psi^{-1}(r^{-d-2\gamma_k})}{\Phi^{-1}(r^{-d-2\gamma_k})} = \frac{(\Psi^{-1}\circ\Phi)\big(\frac{M_{b,k}f(x)}{C_0\|b\|_{BMO_k}\|f\|_{L_{\Phi,k}}}\big)}{\frac{M_{b,k}f(x)}{C_0\|b\|_{BMO_k}\|f\|_{L_{\Phi,k}}}}.$$

Therefore, we get

$$J_0(x) + J_1 + J_2 \le C_1\|b\|_{BMO_k}\|f\|_{L_{\Phi,k}}(\Psi^{-1}\circ\Phi)\left(\frac{M_{b,k}f(x)}{C_0\|b\|_{BMO_k}\|f\|_{L_{\Phi,k}}}\right).$$

Let C_0 be as in (4.4). Consequently by Theorem 6 and (2.4) we have

$$\begin{aligned}\int_B \Psi\left(\frac{J_0(x)+J_1+J_2}{C_1\|b\|_{BMO_k}\|f\|_{L_{\Phi,k}}}\right) h_k^2(x)dx &\le \int_B \Phi\left(\frac{M_{b,k}f(x)}{C_0\|b\|_{BMO_k}\|f\|_{L_{\Phi,k}}}\right) h_k^2(x)dx\\ &\le \int_{\mathbb{R}^n} \Phi\left(\frac{M_{b,k}f(x)}{\|M_{b,k}f\|_{L_{\Phi,k}}}\right) h_k^2(x)dx \le 1,\end{aligned}$$

i.e.

$$\|J_0(\cdot) + J_1 + J_2\|_{L_{\Psi,k}(B)} \lesssim \|b\|_{BMO_k}\|f\|_{L_{\Phi,k}}. \tag{4.8}$$

By (4.3), (2) and condition (4.6), we also get

$$\begin{aligned}\|J_3\|_{L_{\Psi,k}(B)} &= \left\|\sup_{t>2r}\frac{1}{|B(x_0,t)|_k^{1-\frac{\alpha}{d+2\gamma_k}}}\int_{B(x_0,t)}|b(\cdot)-b_B||f(y)|\,h_k^2(y)dy\right\|_{L_{\Psi,k}(B)}\\ &\approx \|b(\cdot)-b_B\|_{L_{\Psi,k}(B)}\sup_{t>2r} t^{\alpha-d-2\gamma_k}\int_{B(x_0,t)}|f(y)|\,h_k^2(y)dy\\ &\lesssim \frac{\|b\|_{BMO_k}}{\Psi^{-1}\big(|B|_k^{-1}\big)}\sup_{t>2r}\Phi^{-1}(|B(x_0,t)|_k^{-1})t^{\alpha}\|f\|_{L_{\Phi,k}(B(x_0,t))}\\ &\lesssim \frac{\|b\|_{BMO_k}}{\Psi^{-1}\big(|B|_k^{-1}\big)}\|f\|_{L_{\Phi,k}}\sup_{t>2r} t^{\alpha}\Phi^{-1}(|B(x_0,t)|_k^{-1})\\ &\lesssim \|b\|_{BMO_k}\,\|f\|_{L_{\Phi,k}}.\end{aligned}$$

Consequently, we have

$$\|J_3\|_{L_{\Psi,k}(B)} \lesssim \|b\|_{BMO_k}\,\|f\|_{L_{\Phi,k}}. \tag{4.9}$$

Combining (4.8) and (4.9), we get

$$\|M_{b,\alpha,k}f\|_{L_{\Psi,k}(B)} \lesssim \|b\|_{BMO_k}\|f\|_{L_{\Phi,k}}. \tag{4.10}$$

By taking supremum over B in (4.10), we get

$$\|M_{b,\alpha,k} f\|_{L_{\Psi,k}} \lesssim \|b\|_{BMO_k} \|f\|_{L_{\Phi,k}},$$

since the constants in (4.10) don't depend on x_0 and r.

2. We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 8, we have $r_0^{\alpha}|b(x) - b_{B_0}| \leq C M_{b,\alpha,k}\chi_{B_0}(x)$. Therefore, by Lemmas 1 and 6

$$r_0^{\alpha} \lesssim \frac{\|M_{b,\alpha,k}\chi_{B_0}\|_{L_{\Psi,k}(B_0)}}{\|b(\cdot) - b_{B_0}\|_{L_{\Psi,k}(B_0)}} \lesssim \Psi^{-1}(|B_0|_k^{-1})\|M_{b,\alpha,k}\chi_{B_0}\|_{L_{\Psi,k}(B_0)}$$
$$\lesssim \Psi^{-1}(|B_0|_k^{-1})\|M_{b,\alpha,k}\chi_{B_0}\|_{L_{\Psi,k}} \lesssim \Psi^{-1}(|B_0|_k^{-1})\|\chi_{B_0}\|_{L_{\Phi,k}} \lesssim \frac{\Psi^{-1}(r_0^{-d-2\gamma_k})}{\Phi^{-1}(r_0^{-d-2\gamma_k})}.$$

Since this is true for every $r_0 > 0$, we are done.

3. The third statement of the theorem follows from the first and second parts of the theorem. □

If we take $\Phi(t) = t^p$ and $\Psi(t) = t^q$ in Theorem 7 we get the following corollary.

Corollary 3 *Let $1 < p < \infty$, $0 < \alpha < (d + 2\gamma_k)/p$ and $b \in BMO_k(\mathbb{R}^d)$. Then $M_{b,\alpha,k}$ is bounded from $L_{p,k}(\mathbb{R}^d)$ to $L_{q,k}(\mathbb{R}^d)$ if and only if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d+2\gamma_k}$.*

By (2.2) and Theorems 7 and 5 we get the following corollary.

Corollary 4 *Let $0 < \alpha < d + 2\gamma_k$, $b \in BMO_k(\mathbb{R}^d)$, $b^- \in L_\infty(\mathbb{R}^d, h_k^2(x)dx)$ and Φ, Ψ be Young functions with $\Phi \in \nabla_2 \cap \mathcal{Y}$ and $\Psi \in \Delta_2$. Let also the condition (4.6) is satisfied. Then the operator $[b, M_{\alpha,k}]$is bounded from $L_{\Phi,k}(\mathbb{R}^d)$ to $L_{\Psi,k}(\mathbb{R}^d)$.*

The following theorem is valid.

Theorem 8 *Let $0 < \alpha < d + 2\gamma_k$, $b \in L_{1,k}^{\mathrm{loc}}(\mathbb{R}^d)$ and Φ, Ψ be Young functions with $\Phi \in \mathcal{Y}$.*

1. *If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$ and the condition (4.6) holds, then the condition $b \in BMO_k(\mathbb{R}^d)$ is sufficient for the boundedness of $M_{b,\alpha,k}$ from $L_{\Phi,k}(\mathbb{R}^d)$ to $L_{\Psi,k}(\mathbb{R}^d)$.*
2. *If $\Psi^{-1}(t) \lesssim \Phi^{-1}(t)t^{-\alpha/(d+2\gamma_k)}$, then the condition $b \in BMO_k(\mathbb{R}^d)$ is necessary for the boundedness of $M_{b,\alpha,k}$ from $L_{\Phi,k}(\mathbb{R}^d)$ to $L_{\Psi,k}(\mathbb{R}^d)$.*
3. *If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$, $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\alpha/(d+2\gamma_k)}$ and the condition (4.7) holds, then the condition $b \in BMO_k(\mathbb{R}^d)$ is necessary and sufficient for the boundedness of $M_{b,\alpha,k}$ from $L_{\Phi,k}(\mathbb{R}^d)$ to $L_{\Psi,k}(\mathbb{R}^d)$.*

Proof

1. The first statement of the theorem follows from the first part of the Theorem 7.
2. We shall now prove the second part. Suppose that $M_{b,\alpha,k}$ is bounded from $L_{\Phi,k}(\mathbb{R}^d)$ to $L_{\Psi,k}(\mathbb{R}^d)$. Choose any ball $B = B(x, r)$ in $\mathbb{R}^d$, by (2.3)

$$\begin{aligned}
&\frac{1}{|B|_k}\int_B |b(y)-b_B|\, h_k^2(y)dy \\
&= \frac{1}{|B|_k}\int_B \left|\frac{1}{|B|_k}\int_B (b(y)-b(z))\, h_k^2(z)dz\right| h_k^2(y)dy \\
&\le \frac{1}{|B|_k^2}\int_B\int_B |b(y)-b(z)|\, h_k^2(z)dz\, h_k^2(y)dy \\
&= \frac{1}{|B|_k^{1+\frac{\alpha}{d+2\gamma_k}}}\int_B \frac{1}{|B|_k^{1-\frac{\alpha}{d+2\gamma_k}}}\int_B |b(y)-b(z)|\chi_B(z)\, h_k^2(z)dz\, h_k^2(y)dy \\
&\le \frac{1}{|B|_k^{1+\frac{\alpha}{d+2\gamma_k}}}\int_B M_{b,\alpha,k}\big(\chi_B\big)(y)\, h_k^2(y)dy \\
&\le \frac{2}{|B|_k^{1+\frac{\alpha}{d+2\gamma_k}}}\,\|M_{b,\alpha,k}\big(\chi_B\big)\|_{L_\Psi(B)}\|1\|_{L_{\widetilde{\Psi}}(B)} \\
&\le \frac{C}{|B|_k^{\frac{\alpha}{d+2\gamma_k}}}\,\frac{\Psi^{-1}(|B|_k^{-1})}{\Phi^{-1}(|B|_k^{-1})} \le C.
\end{aligned}$$

Thus $b \in BMO_k(\mathbb{R}^d)$.

3. The third statement of the theorem follows from the first and second parts of the theorem.

□

If we take $\Phi(t) = t^p$ and $\Psi(t) = t^q$ in Theorem 8 we get the following corollary.

Corollary 5 *Let $1 < p < \infty$, $0 < \alpha < (d+2\gamma_k)/p$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d+2\gamma_k}$. Then $M_{b,\alpha,k}$ is bounded from $L_{p,k}(\mathbb{R}^d)$ to $L_{q,k}(\mathbb{R}^d)$ if and only if $b \in BMO_k(\mathbb{R}^d)$.*

Corollary 6 *Let $1 < p < \infty$, $0 < \alpha < (d+2\gamma_k)/p$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d+2\gamma_k}$, $b \in BMO_k(\mathbb{R}^d)$ and $b^- \in L_{\infty,k}(\mathbb{R}^d)$. Then $[b, M_{\alpha,k}]$ is bounded from $L_{p,k}(\mathbb{R}^d)$ to $L_{q,k}(\mathbb{R}^d)$.*

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